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Ramsey-Friedman optimality with banking time

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Abstract
This chapter conducts a Ramsey analysis within an endogenous growth cashin-advance economy with policy commitment. Credit and money are alternative payment mechanisms that act as inputs into the household production of exchange. The credit is produced with a diminishing returns technology with Inada conditions that implies along the balanced-growth path a degree one homogeneity of effective banking time. This tightens the restrictions found within shopping time economies while providing a production basis for the Ramsey-Friedman optimum that suggests a special case of Diamond and Mirrlees (1971).

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6 Ramsey-Friedman optimality with banking time*

Max Gillman and Oleg Yerokhin

Summary

This chapter conducts a Ramsey analysis within an endogenous growth cash-in-advance economy with policy commitment. Credit and money are alternative payment mechanisms that act as inputs into the household production of exchange. The credit is produced with a diminishing returns technology with Inada conditions that implies along the balanced-growth path a degree one homogeneity of effective banking time. This tightens the restrictions found within shopping time economies while providing a production basis for the Ramsey-Friedman optimum that suggests a special case of Diamond and Mirrlees (1971).

6.1 Introduction

Homogeneity of the shopping time function in Correia and Teles (1999) is necessary for Ramsey (1927) optimality of the Friedman (1969) rule. This second-best Friedman optimum is an interesting result in that it occurs in one of the most standard exchange economies in use today. However the required homogeneity of the arguments in the shopping time function is difficult to interpret since this is a general transactions cost technology, involving the input of the consumer's shopping time as derived from some combination of real money and consumption. One interpretation is supplied by Lucas (2000). In specifying the shopping time model, he chooses a functional form that makes shopping time inversely proportional to the consumption velocity of real money demand (see also Canzoneri and Diba, 2005). This implies a money demand interest elasticity of – 0.5 as in Baumol (1952) and a unitary income elasticity, while implying a shopping time function that is homogenous of degree one in real money and consumption goods. More generally, Correia and Teles (1999) do not impose a unitary income elasticity of money demand and find that any degree of homogeneity is sufficient for the Friedman rule to be Ramsey optimal, although here the implications for the underlying money demand function are not drawn out.

This chapter contributes a different approach that offers a new derivation

and interpretation of the homogeneity result. Alternatively it can be viewed as a more restrictive approach that focuses like Lucas (2000) on the resulting money demand function. The model imposes restrictions on the transactions technology by assuming that credit is produced with diminishing returns to labor. It is assumed that exchange credit is produced using labor time (or "banking time") and goods in a Cobb-Douglas fashion, where the credit serves as a costly way to buy goods without using interest-foregoing money. This results in a money demand function with an interest elasticity similar to Cagan's in that it rises with the inflation rate; and it has a unitary consumption elasticity.

The credit production specification also yields a restriction equivalent to homogeneity of degree one on effective time used in exchange in a way that is comparable to a special case of a shopping time function. The credit production specification is partly restricted by the need of its endogenous growth setting to have all variables in the economy grow at the same rate on the balanced-growth path (BGP) equilibrium. This means that money and consumption must grow at the same rate, and consumption velocity must be stable, giving a unitary consumption elasticity of money demand. This in turn restricts the homogeneity on the time spent in transactions.

The key necessary condition for Ramsey (1927) optimality of the Friedman rule is that the marginal productivity of the banking time in producing credit must be driven to infinity. At this point, with only an assumption of diminishing returns in producing credit per unit of consumption, credit production is indeed zero and only money is used by the consumer for exchange. In contrast, the Ramsey (1927) optimality condition within the shopping time economy requires that there is "satiation" of real money balances so that there is no use of shopping time required once this particular satiation level of money demand is reached. This also involves the additional assumption that the change in shopping time with respect to money is equal to zero at that point, a differentiability required for the Ramsey optimum. The banking time model instead substitutes zero credit use for the satiation point and substitutes a diminishing marginal product of labor in credit production with Inada conditions for the differentiability of the shopping time function.

The resulting Ramsey (1927) optimality of the Friedman (1969) optimum can be interpreted as a special case of Diamond and Mirrlees (1971): credit is specified as an alternative input into producing exchange, along with money, making it an intermediate good within a Becker (1975) household production economy. The consumer needs not only the good, but also the exchange means to get the good, either money or credit.[1] This is why the shadow price of consumption contains a shadow goods cost component (one) plus a shadow exchange cost component (a weighted average of the cost of using cash and of using credit). The shadow costs reflect a Becker (1975)- like interpretation of money and credit as inputs, and this provides the second-best intuition: intermediate goods with CRS production functions are not to be taxed because it distorts the production margins (the input allocations) as well as the consumption margins (goods versus leisure), as long as the goods output is also CRS produced (Diamond and Mirrlees 1971). The Ramsey-Diamond-Mirrless result in the economy implies that money as an input to exchange should not be taxed when other taxes are available that do not distort the exchange production margin; otherwise the efficient production of exchange is needlessly distorted towards wasteful inflation-tax avoiding credit use. Zero credit production is second-best optimal because it avoids unnecessary distortions to production efficiency.

6.2 The "banking time" economy

6.2.1 The consumer problem

The representative consumer's time period t utility function depends on consumption goods and leisure, and is given by $u(c_t, x_t)$, with the assumed Inada conditions with respect to c_t and x_t. Discounted by the time preference rate $\rho \in (0, 1)$ the utility stream is

$$\int_0^\infty e^{-\rho t} u(c_t, x_t)\, dt. \tag{1}$$

The consumer divides an endowment of 1 unit of time between working to produce goods output, l_t, working to produce credit, l_{dt}, investing in human capital production, l_{ht}, and taking leisure, x_t. The allocation of time constraint can be written as

$$1 = l_t + l_{dt} + l_{ht} + x_t. \tag{2}$$

6.2.1.1 Production technology

Consumption with goods and exchange Consider a Becker (1975)-type household production economy as extended to include exchange activity as part of household production, and also including human capital (Lucas 1988b). The consumer engages in household production of exchange, using money and credit, and of the consumption good using goods output and exchange. The good that the agent consumes is the aggregate consumption good, denoted c_t. This is produced using the aggregate output y_t that is devoted to consumption goods, denoted by y_{ct}, and an amount of exchange that is needed to purchase the good, denoted by y_{et}. Note here that only consumption goods are assumed to require exchange; capital and labor markets do not require exchange. Let the production of the consumption good be Leontieff in terms of the goods output and the exchange. Whereas Aiyagari, Braun, and Eckstein (1998) use a Leontieff technology to produce the credit good at each store of a continuum of stores, here the approach is extended by having an aggregate good combined with exchange, either cash or credit, in Leontieff fashion to produce the Becker (1975)-type consumption good:

$$c_t = \min(y_{ct}, y_{et}). \tag{3}$$

Only the efficient frontier of the Leontieff production of the consumption good will be utilized, this being a ray from the origin in isoquant space, if the relative price of the output of goods to the exchange means for goods is between zero and infinity. Here the assumption is that the slope is one. This means simply that the amount of goods bought corresponds directly to the amount of money or credit paid for the goods, in a one-to-one fashion.[2] This implies that along the ray

$$c_t = y_{ct}; \tag{4}$$

$$c_t = y_{et}. \tag{5}$$

The production of output y_t is a standard constant returns to scale function in capital, k_t, and effective labor, the human capital, h_t, factored by the labor supply, l_t:

$$y_t = f(k_t, l_t h_t) = A k_t^{\alpha} (l_t h_t)^{1-\alpha}. \tag{6}$$

The production of human capital is given by the function $H(\cdot)$ that has as its only argument l_{ht}:

$$\dot{h}_t = h_t H(l_{ht}). \tag{7}$$

It is assumed that $H'(l_{ht}) > 0$, and $H''(l_{ht}) < 0$.

The production of exchange requires inputs of real money balances and/or real credit. Denote the real money balances as $m_t \equiv M_t/P_t$, with M_t denoting the nominal money stock, and P_t denoting the price of the aggregate consumption good. And let real credit be denoted by d_t. The production of exchange is assumed to be homogeneous of degree one in m_t and d_t. In general it given as

$$y_{et} = f_e(m_t, d_t). \tag{8}$$

Specifically, assume that real money and credit are perfect substitutes, so that

$$y_{et} = m_t + d_t. \tag{9}$$

Credit The credit technology is a costly self-produced means of purchasing goods instead of using money. This might be thought of as an abstraction from a world with payment uncertainty, where for example, the agent produces information about his purchase and payment history that enables credit to be issued just as a credit agency might. Here there is not a decentralized credit market, but rather the representative agent simply acts in part as a bank, producing what can be called exchange credit.[3]

The specification is that the effective labor per unit of consumption produces the share of credit in total purchases with a diminishing marginal productivity. In particular, from equations (4), (5), and (9), $d_t/c_t = 1 - (m_t/c_t)$. Define the share

$$a_t \equiv m_t/c_t. \tag{10}$$

Then $d_t = c_t(1 - a_t)$ is the total credit used. Specify the production of this credit, with $\gamma \in (0, 1)$, as

$$d_t = c_t A_{dt} (l_{dt} h_t / c_t)^{\gamma} = A_d (l_{dt} h_t)^{\gamma} c_t^{1-\gamma}, \tag{11}$$

or in terms of the share a_t:

$$1 - a_t = A_d (l_{dt} h_t / c_t)^{\gamma} \tag{12}$$

The diminishing returns technology implies that the marginal cost per unit of consumption is an upward sloping curve that depends on the parameter γ. This marginal cost (MC_t) can be defined as the marginal factor cost divided by the marginal factor product, or, with w_t denoting the marginal labor cost, $MC_t \equiv (w_t/\gamma)\, A_d^{-1/\gamma}\, (d_t/c_t)^{(1-\gamma)/\gamma}$ (this definition instead can be derived from the BGP equilibrium conditions below, as in equation (49), where $R_t = MC_t$). With $\gamma = 0.5$, this marginal cost curve slopes upward with a straight line; with $\gamma < 0.5$ it exhibits an upward sloping convex marginal cost curve that entails an increasing marginal cost as output of credit per unit of consumption increases. This (0, 0.5) range is the most plausible for γ since it produces the typically shaped marginal cost curve; for the (0.5, 1) range the marginal costs rise at a decreasing rate.

6.2.1.2 Total income

The consumer buys and sells nominal government bonds, denoted by B_t, which earn the nominal interest rate of R_t. The change over time in the real bond purchases is $\dot{B}_t/P_t$ and the real value of the interest is $R_t\, B_t/P_t$. This net purchase of bonds, plus the consumer's real goods purchases y_{ct}, equal to c_t by equation (4), plus the capital investment, denoted by $\dot{k}_t$, and investment in real money $\dot{M}_t/P_t$ are equal to the after tax return to labor and capital rentals plus the bond income. With W_t and r_t denoting the rental prices of labor and capital, denote the after tax real wage and interest rental rates as

$$\tilde{w}_t \equiv (1 - \tau_t^l) w_t, \tag{13}$$

$$\tilde{r}_t \equiv (1 - \tau_t^k) r_t. \tag{14}$$

The consumer budget constraint is

$$c_t + \dot{k}_t + \dot{M}_t/P_t + \dot{B}_t/P_t = \tilde{w}_t l_t h_t + \tilde{r}_t k_t + R_t B_t/P_t. \tag{15}$$

The money investment can be written as

$$\dot{M}_t/P_t = \dot{m}_t + m_t \pi_t, \tag{16}$$

and inserted into equation (15) to give

$$c_t + \dot{k}_t + \dot{m}_t + m_t \pi_t + \dot{B}_t/P_t = \tilde{w}_t l_t h_t + \tilde{r}_t k_t + R_t B_t/P_t. \tag{17}$$

The consumer problem could now be stated in a Hamiltonian form (see for example Turnovsky (1997)) as the maximization of utility (1) subject to the allocation of time constraint (2), the human capital investment constraint (7), the income constraint (17) and the money and credit constraints (10), (11) and (12). The choice variables would be m_t, d_t, and a_t plus all of the time allocations, the goods consumption, and the physical and human capital levels. A reduced set of constraints results by eliminating a_t by combining the money and credit constraints into one constraint of $c_t = d_t + m_t$. The credit output d_t can be eliminated by substituting in from the credit technology equation (11) so that the constraints are now $c_t A_d (l_{dt} h_t / c_t)^{\gamma} = c_t + m_t$, plus the human capital investment, allocation of time, and income constraints. A further reduced set of constraints, of human capital investment, allocation of time, and income, can result by eliminating the banking time l_{dt} by solving for it in terms of c_t, m_t, and h_t and substituting this into the allocation of time constraint (see equation (31) below).

Note that in the Hamiltonian, differentiating with respect to the physical capital level and the nominal bond level B_t, yields the Fisher equation

$$R_t = \pi_t + \tilde{r}_t. \tag{18}$$

This gets suppressed when using the wealth constraint approach below with Ricardian equivalence.

6.2.2 The goods producer problem

The goods producer rents labor and capital from the consumer, taking the competitive real prices of labor and capital as given. The firm's first-order conditions, using equation (6), are

$$w_t = (1-a)\left(\frac{k_t}{l_t h_t}\right)^a = f_{lh}(k_t, l_t h_t), \tag{19}$$

$$r_t = a\left(\frac{k_t}{l_t h_t}\right)^{a-1} = f_k(k_t, l_t h_t), \tag{20}$$

and the CRS production function implies that there are zero profits.

6.2.3 Government budget constraint

The government has no access to lump sum taxes and finances its expenditure g_t partly with flat proportional taxes on labor and capital income. With these tax rates at time t denoted by τ_t^l and τ_t^k, this real tax income is $\tau_t^k r_t k_t + \tau_t^l w_t l_t h_t$. Added to this is the net proceeds of new bond issues $(\dot{B}_t - R_t B_t)/P_t$, and proceeds from new money printing $(M_{t+1} - M_t)/P_t$, where the nominal money supply is assumed to exogenously grow at a constant rate σ through open market operations, and where the consumer is already given the initial stock $M_0 > 0$. The government budget constraint is given by

$$\tau_t^k r_t k_t + \tau_t^l w_t l_t h_t + \left(\dot{B}_t - R_t B_t\right)/P_t + \dot{M}_t/P_t = g_t. \tag{21}$$

6.2.4 Resource constraint

Writing out the consumer's income constraint (17) by using that $\tilde{w}_t \equiv (1-\tau_t^l)w_t$, and $\tilde{r}_t \equiv (1-\tau_t^k)r_t$ (equations 13 and 14), so that

$$c_t + \dot{k}_t + \dot{M}_t/P_t + \dot{B}_t/P_t = (1-\tau_t^l)w_t l_t h_t + (1-\tau_t^k)r_t k_t + R_t B_t/P_t,$$

and substituting in for $\tau_t^k r_t k_t + \tau_t^l w_t l_t h_t$ from the government budget constraint (21), gives that

$$c_t + \dot{k}_t + g_t = w_t l_t h_t + r_t k_t.$$

Using the CRS property of goods production, whereby

$$w_t l_t h_t + r_t k_t = A k_t^a (l_t h_t)^{1-a},$$

this then reduces to the resource constraint of

$$c_t + \dot{k}_t + g_t = A k_t^a (l_t h_t)^{1-a}. \tag{22}$$

6.3 Equilibrium

6.3.1 The wealth constraint

A formulation convenient for the Ramsey (1927) problem is to construct the wealth constraint from the income flow constraint (17). Define real wealth, denoted as W_t, by the sum of physical capital and the real money stock:

$$W_t = k_t + m_t + B_t/P_t. \tag{23}$$

Then from equations (17), (23), and using the Fisher identity in (18) that $R_t = \pi_t + \tilde{r}_t$, it follows that

$$\dot{W}_t = \tilde{r}_t(k_t + m_t + B_t/P_t) + \tilde{w}_t h_t l_t - c_t - R_t m_t. \tag{24}$$

Note that it is assumed that $R_t \geq 0$ so that the wealth constraint is not unbounded (see for example Ljungqvist and Sargent (2000)).

Given the initial period m_0 and k_0, integrating over the infinite horizon, and imposing the transversality conditions,

$$\lim_{t\to\infty} m_t e^{-\int_0^t \tilde{r}_s ds} = 0; \tag{25}$$

$$\lim_{t\to\infty} k_t e^{-\int_0^t \tilde{r}_s ds} = 0; \tag{26}$$

$$\lim_{t\to\infty} (B_t/P_t)\, e^{-\int_0^t \tilde{r}_s ds} = 0; \tag{27}$$

the wealth constraint (see Appendix 6.A.1) is

$$\int_0^\infty e^{-\int_0^t \tilde{r}_s ds}\left[c_t + R_t m_t - \tilde{w}_t h_t l_t\right] dt = m_0 + k_0 + B_0/P_0. \tag{28}$$

Constraint (28) is a dynamic version of the income constraint of Mulligan and Sala-I-Martin (1997) (see equation 2, p.7).

The consumer problem can be stated as the maximization of utility (equation 1) subject to equations (2), (7), (10), (12) and (28), the allocation of time, human capital investment, money, credit and wealth constraints, with respect to c_t, x_t, l_t, l_{ht}, l_{dt}, a_t, m_t, k_t, and h_t.

6.3.2 Definition of equilibrium

The competitive equilibrium consists of a time path for the allocation $\{yct, y_{et}, d_t, c_t, x_t, l_t, l_{ht}, l_{dt}, m_t, k_t, h_t\}_{t=0}^{\infty}$ given the input prices $\{w_t, r_t\}_{t=0}^{\infty}$, tax rates $\{\tau_t^l, \tau_t^k\}_{t=0}^{\infty}$, government spending $\{g_t\}_{t=0}^{\infty}$, and the initial period k_0, M_0, B_0 and

P_0 (normalized to one), such that $\{c_t, x_t, l_t, l_{ht}, m_t, k_t, h_t, B_t/P_t\}_{t=0}^{\infty}$ maximizes (1) subject to constraints (2), (7), (10), (12) and (28), and such that $\{\tau_t^l, \tau_t^k\}_{t=0}^{\infty}$, and $\{k_t, l_t, h_t, r_t, w_t\}_{t=0}^{\infty}$, satisfy the constraints (6), (19), (20), (21) and (22), and that constraints (4), (5), (9), and (11) are satisfied.

6.3.3 Characterization of equilibrium

The effective labor in credit production, which in equilibrium can be thought of as the derived demand, can be solved from equations (10) and (12) as

$$l_{dt}\, h_t = (A_d)^{-1/\gamma}\, c_t\,]1 - (m_t/c_t)]^{1/\gamma}. \tag{29}$$

Equation (29) is mathematically analogous to a special case of the McCallum and Goodfriend (1987) shopping time economy (Walsh 1998) as extended to endogenous growth. But here instead the concept is banking time that is used to produce an intermediate good, credit, that in turn is combined with money to produce another intermediate good, exchange, which finally is combined in Leontieff fashion with goods output to produce consumption goods.

Solving for l_{dt} and defining it as $b(c_t, m_t, h_t)$,

$$l_{dt} = \left(A_d^{-1/\gamma}\, c_t\, [1 - (m_t/c_t)]^{1/\gamma}\right)/h_t \equiv b(c_t, m_t, h_t), \tag{30}$$

where $b_c > 0$, $b_m < 0$, $b_h < 0$, this raw *banking time* can be substituted directly into the allocation of time constraint (2):

$$1 = l_t + b(c_t, m_t, h_t) + l_{ht} + x_t, \tag{31}$$

while $l_{dt}h_t$ is the effective banking time. The function $b(c_t, m_t, h_t)$ of equation (30) exhibits homogeneity of degree one in c_t and m_t as in equation (29), and exhibits homogeneity of degree zero (HD0) in its three arguments.

The present value Hamiltonian for the consumer problem can then be written as

$$\begin{aligned}\mathcal{H} = {} & e^{-\rho t}u(c_t, x_t) + \mu_t h_t H(l_{ht}) + \theta_t\,[1 - x_t - b\,(c_t, m_t, h_t) - l_t - l_{ht}] \\ & + \lambda\left[m_0 + k_0 + B_0/P_0 + \int_0^{\infty} e^{-\int_0^t \tilde{r}_s ds}\,(\tilde{w}_t h_t l_t - c_t - R_t m_t)\, dt\right].\end{aligned} \tag{32}$$

The first-order conditions are

$$e^{-\rho t}u_c(c_t, x_t) - \lambda e^{-\int_0^t \tilde{r}_s ds} - \theta_t b_c(c_t, m_t, h_t) = 0; \tag{33}$$

$$e^{-\rho t}u_x(c_t, x_t) - \theta_t = 0; \tag{34}$$

$$\lambda e^{-\int_0^t \tilde{r}_s ds}\,\tilde{w}_t h_t - \theta_t = 0; \tag{35}$$

$$\mu_t h_t H'(l_{ht}) - \theta_t = 0; \tag{36}$$

$$-\lambda e^{-\int_0^t \tilde{r}_s ds}\, R_t - \theta_t b_m(c_t, m_t, h_t) = 0; \tag{37}$$

$$\mu_t H(l_{ht}) + \lambda e^{-\int_0^t \tilde{r}_s ds}\,\tilde{w}_t l_t - \theta_t b_h(c_t, m_t, h_t) = -\,\dot{\mu}_t. \tag{38}$$

Combining equations (35) and (38) to get

$$\mu_t H(l_{ht}) + \lambda e^{-\int_0^t \tilde{r}_s ds}\,\tilde{w}_t[l_t - h_t b_h(c_t, m_t, h_t)] = -\,\dot{\mu}_t; \tag{39}$$

multiplying through by h_t and substituting in equation (7) gives

$$\dot{\mu}_t h_t + \mu_t \dot{h}_t = -\,\lambda e^{-\int_0^t \tilde{r}_s ds}\,\tilde{w}_t h_t[l_t - h_t b_h(c_t, m_t, h_t)]. \tag{40}$$

This can be written as

$$\frac{d}{ds}(\mu_s\, h_s) = -\,\lambda e^{-\int_0^t \tilde{r}_\xi d\xi}\,\tilde{w}_s h_s[l_s - h_s b_h(c_s, m_s, h_s)]. \tag{41}$$

Integrating both sides from t to ∞ and imposing the transversality condition

$$\lim_{s \to \infty} \mu_s h_s = 0, \tag{42}$$

gives that

$$\mu_t h_t = \lambda e^{-\int_0^t \tilde{r}_\xi d\xi} \int_t^{\infty} e^{-\int_t^s \tilde{r}_\xi d\xi}\,\tilde{w}_s h_s[l_s - h_s b_h(c_s, m_s, h_s)]. \tag{43}$$

Substituting in equations (35) and (36), and using the fact that

$$-\,h_t b_h\,(c_t, m_t, h_t) = b_c(c_t, m_t, h_t)c_t + b_m(c_t, m_t, h_t)m_t = b(c_t, m_t, h_t), \tag{44}$$

gives the Becker (1975)-type [p. 68, equation (63)] margin of human capital accmulation, stated as

$$\tilde{w}_t h_t = H'(l_{ht}) \int_t^{\infty} e^{-\int_t^s \tilde{r}_\xi d\xi}\,\tilde{w}_s h_s[l_s + b(c_s, m_s, h_s)]ds. \tag{45}$$

Equation (45) is the Euler equation for the motion of human capital in the economy with banking time. The left-hand side is the workers earnings if a unit of time is spent in the production of goods. The right-hand side is the product of two terms: the percentage increase in human capital if a unit of time is spent in human capital accumulation, and the discounted value of increased earnings flow that this additional human capital will yield. Alternatively this condition can be written as

$$e^{-\rho t}\, u_x(c_t, x_t) = \lambda\left(e^{-\int_0^t \tilde{r}_\xi d\xi}\, H'(l_{ht}) \int_t^\infty e^{-\int_t^s \tilde{r}_\xi d\xi}\, \tilde{w}_s h_s\,[l_s + b(c_s, m_s, h_s)]\, ds\right), \tag{46}$$

defining the margin of leisure time versus time spent in human capital accumulation. The left-hand side is the utility value of a unit of time devoted to leisure in period t from the point of view of period 0. The right-hand side is the same value of time in human capital accumulation as in equation (45), now discounted back to period zero and converted to its utility value through multiplication by the shadow value of wealth.

From equations (33) and (35), the intertemporal consumption marginal rate of substitution between dates 0 and t is

$$\frac{e^{-\rho t}\, u_c(c_t, x_t)}{u_c(c_0, x_0)} = \frac{e^{-\int_0^t \tilde{r}_s ds}\,[1 + \tilde{w}_t h_t b_c(c_t, m_t, h_t)]}{1 + \tilde{w}_0 h_0 b_c(c_0, m_0, h_0, B_0/P_0)}. \tag{47}$$

Equation (33), (34) and (35) imply that the marginal rate of substitution between consumption and leisure is

$$\frac{u_c(c_t, x_t)}{u_x(c_t, x_t)} = \frac{1 + \tilde{w}_t h_t b_c(c_t, m_t, h_t)}{\tilde{w}_t h_t}, \tag{48}$$

being equal to the ratio of the shadow prices of consumption and leisure, comparable to, for example, Walsh's (1998) shopping time model, where 1 is the goods cost and $\tilde{w}_t h_t b_c(c_t, m_t, h_t)$ the exchange cost. Since $\tilde{w}_t h_t b_c(c_t, m_t, h_t) < \infty$, there is a solution at the corner of the square Leontieff isoquant in equation (3). In particular, the slope along the Leontieff isoquant is either zero or infinity. Thus if there is a relative cost of goods versus exchange which is between zero and infinity, then this guarantees that the slope of the isocost line is between zero and infinity and touches the isoquant at its corner. Here being in a "corner" is good. It produces an interior solution that guarantees that the consumer indeed chooses to combine an equal amount of goods and exchange in order to "produce" the consumption good from these two inputs.

Also note the alternative interpretation of the shadow exchange cost of goods, $\tilde{w}_t h_t b_c$. It can be shown that in equilibrium $\tilde{w}_t h_t b_c = a_t R_t + (1 - a_t)\gamma R_t$. The term $a_t R_t + (1 - a_t)\gamma R_t$ is a weighted average of the average costs of money and credit. The average cost of credit γR_t is less than the average cost of money R_t since $\gamma < 1$. This means that although the marginal cost of credit is equal to R_t in equilibrium, its average cost is less and so the consumer saves by using credit.

To see that the marginal cost of credit is equal to the nominal interest rate, use equations (35) and (37) to write

$$\tilde{w}_t h_t b_m = R_t. \tag{49}$$

This is the analogue to the original equilibrium condition in Baumol (1952) from minimizing the costs of using money or going to the bank. The condition (49) similarly equalizes the marginal cost of the alternative exchange means. This follows by using equations (10), (11), (12) and (30) to show that $b_m = 1/[\partial d_t/\, \partial(l_{dt})]$, so that $\tilde{w}_t h_t b_m = \tilde{w}_t h_t/[\partial d_t/\partial(l_{dt})]$. This latter term is the marginal factor cost divided by the marginal factor product, which by microeconomic theory is equal to the marginal cost of the output d_t. This relation implies that R_t is equal to the marginal cost of credit.

6.3.4 Money and banking

The condition (49) that equalizes the marginal costs of exchange, along with equations (10) (11), and (12), also yields the solution for a_t, the money demand function, and the consumption velocity:

$$R_t = \tilde{w}_t h_t b_m(c_t, m_t, h_t) = \tilde{w}_t A_d^{1/\gamma}[1 - (m_t/c_t)]^{(1/\gamma)-1}/\gamma; \tag{50}$$

$$a_t = 1 - \left[A_d^{1/(1-\gamma)}\,(R_t\gamma/\tilde{w}_t)^{\gamma/(1-\gamma)}\right]; \tag{51}$$

$$m_t = c_t\left(1 - \left[A_d^{1/(1-\gamma)}\,(R_t\gamma/\tilde{w}_t)^{\gamma/(1-\gamma)}\right]\right). \tag{52}$$

The consumption velocity is $c_t/m_t = 1/\{1 - [(R_t\gamma)/(\tilde{w}_t A_d)]^{\gamma/(1-\gamma)}\}$; it is constant on the balanced-growth path. This results because the wage rate of effective labor, $w_t = (1 - \alpha)\left(\frac{k_t}{l_t h_t}\right)^{\alpha}$, depends on the capital to effective labor ratio; k_t and h_t grow at the same rate on the balanced-growth path, and the labor share l_t is constant on the balanced-growth path. Since $\tilde{w}_t \equiv (1 - \tau_s^l)w_t$, and given that the labor and capital tax rates are also constant on the balanced-growth path, so $\tilde{w}_t$ is also constant. With $\tilde{r}_t$ constant as well, the nominal interest rate is constant on the balanced-growth path, and so is the consumption velocity. This also gives a unitary consumption elasticity.

The banking time is also constant on the balanced-growth path. From equation (30), $l_{dt} = A_d^{-1/\gamma}(c_t/h_t)[1 - (m_t/c_t)]^{1/\gamma}$. With c_t and h_t growing at the same rate on the balanced-growth path, and with m_t/c_t also constant, so is the banking time. These balanced-growth conditions also make the banking time homogeneous of degree one with respect to m_t and c_t.

The interest elasticity of m_t/c_t, or of the money demand normalized by consumption, is denoted by η_a^R, and given by $\eta_a^R = -[\gamma/(1-\gamma)](1 - a_t)/a_t$. As the interest rises, the credit to cash ratio, $(1 - a_t)/a_t$, rises and the normalized interest elasticity becomes more negative: $\partial \eta_a^R / \partial R_t = -R_t[\gamma/(1-\gamma)]^2(1-a_t)/(a_t)^2 < 0$. Its increasing elasticity with inflation is similar to that in the Cagan (1956) model. Or another way to see the interest elasticity is to write it in terms of the elasticity of substitution between the two inputs money and credit, denoted by ε. Following Gillman and Kejak (2005b), define this as $\in \equiv \left[\partial\left(\frac{ac}{(1-a)c}\right) / \partial\left(\frac{R}{\tilde{w}_t/\gamma A_d^{1/\gamma}}\right)\right]\left[\left(\frac{R}{\tilde{w}_t/\gamma A_d^{1/\gamma}}\right) / \left(\frac{ac}{(1-a)c}\right)\right]$, which is solved as $\in = -[\gamma/(1-\gamma)]/a$. Then with η_a^R denoting the interest elasticity of money demand (not normalized) and η_c^R *denoting the interest elasticity of consumption*, the interest elasticity of money can be written as a sum of the share of the substitute factor, credit, factored by the elasticity of substitution between money and credit, plus a scale effect:

$$\eta_m^R = (1-a) \in + \eta_c^R. \tag{53}$$

And at $R = 0$, the interest elasticity is zero since by equation (49) $l_d = 0$ and $\eta_c^R = [1/(1-\gamma)]\,(\tilde{w}_t l_d h\, c)/(1 + \tilde{w}_t l_d h/c) = 0$ and the share of credit is zero. As the nominal interest rate rises from zero the interest elasticity gradually rises in magnitude from zero.

6.4 The Ramsey optimum

Here the primal approach to optimal taxation (Ljungqvist and Sargent 2000) is used to express time t prices in terms of allocations (see Appendix 6.A.2).

From equations (34), (35), and (37)

$$\tilde{w}_t h_t = \frac{u_x(c_t, x_t)}{u_c(c_t, x_t) - u_x(c_t, x_t) b_c(c_t, m_t, h_t)}. \tag{54}$$

Equations (33), (34), and (37) imply that

$$R_t = \frac{u_x(c_t, x_t) b_m(c_t, m_t, h_t)}{u_x(c_t, x_t) b_c(c_t, m_t, h_t) - u_c(c_t, x_t)}. \tag{55}$$

From equations (34) and (35), and given that λ is constant for all t, it follows that $\lambda = u_x(c_0, x_0)/\tilde{w}_0 h_0$, and that, with equations (33) and (34),

$$e^{-\int_0^t \tilde{r}_s ds} = \frac{\tilde{w}_0 h_0}{u_x(c_0, x_0)} e^{-\rho t}[u_c(c_t, x_t) - u_x(c_t, x_t) b_c(c_t, m_t, h_t)]. \tag{56}$$

Assuming that $\tau_0^k = \tau_0^l = 0$, and using equations (19) and (20), this expression can be written as

$$e^{-\int_0^t f_k(k_s, l_s h_s) ds} = \frac{f_{lh}(k_0, l_0 h_0) h_0}{u_x(c_0, x_0)} e^{-\rho t}[u_c(c_t, x_t) - u_x(c_t, x_t) b_c(c_t, m_t, h_t)]. \tag{57}$$

Substituting equations (54), (55) and (57) into equations (28) and (46), and using the homogeneity properties of credit time function as given in equation (30), the implementability constraints can be derived as

$$\frac{u_x(c_0, x_0)[k_0 + m_0]}{f_{lh}(k_0, l_0 h_0) h_0} = \int_0^\infty e^{-\rho t}\{u_c(c_t, x_t) c_t - u_x(c_t, x_t)[l_t + b(c_t, m_t, h_t)]\}\, dt, \tag{58}$$

$$u_x(c_t, x_t) = H'(l_{ht}) \int_t^\infty e^{-\rho(s-t)} u_x(c_s, x_s)[l_s + b(c_s, m_s, h_s)] ds. \tag{59}$$

Equation (58) is the consumer's budget constraint with prices expressed in terms of allocations. The use of human capital accumulation constraint (59) is motivated by the fact that human capital accumulation occurs outside of the market and cannot be taxed. There is no tax instrument that can be used to make this Euler equation hold for an arbitrary allocation, and consequently it constitutes a constraint on the set of competitive allocations. One way to approach this problem was suggested by Jones, Manuelli, and Rossi (1997), who solve for the Ramsey (1927) plan without including this constraint and then check to see if it is satisfied by the first-order conditions to the planner's problem in the steady-state (see also Ljungqvist and Sargent (2000)). Alternatively, here the constraint is included in the maximization problem explicitly.

The Ramsey (1927) problem can be formulated as the social planner's maximization of the representative agent's utility (1) subject to the implementability constraints (58), (59) and the goods and time resource constraints. The goods resource constraint (22) can be combined with the time constraint (31) to give

$$Ak_t^a([1 - x_t - l_{ht} - b(c_s, m_s, h_s)]h_t)^{1-a} - c_t - \dot{k}_t - g_t = 0. \tag{60}$$

This gives the Ramsey problem of

$$\underset{c_t, x_t, m_t, l_t, l_{ht}, h_t, k_t}{Max} \quad \mathcal{H} = \int_0^\infty u(c_t, x_t) dt \tag{61}$$

$$+\varphi_t\{Ak_t^a([1 - x_t - l_{ht} - b(c_t, m_t, h_t)]h_t)^{1-a} - c_t - \dot{k}_t - g_t\}$$

$$+\,\Phi\left(\frac{u_x(c_0, x_0)[k_0 + m_0 + B_0/P_0]}{f_{lh}(k_0, l_0 h_0) h_0} - \right.$$

$$\left.\int_0^\infty e^{-\rho t}\{u_c(c_t, x_t) c_t - u_x(c_t, x_t)[l_t + b(c_t, m_t, h_t)]\} dt\right)$$

$$+ \Lambda_t \left(u_x(c_t, x_t) - H'(l_{ht}) \int_t^{\infty} e^{-\rho(s-t)} u_x(c_s, x_s)[l_s + b(c_s, m_s, h_s)] ds \right).$$

Lemma 1 *With positive resources, the optimum monetary policy in the endogenous growth economy with a "banking time" specification of the transaction costs function, is satisfied only if*

$$b_m(c_t, m_t, h_t) = 0. \tag{62}$$

proof. The first-order condition of the problem in equation (61) with respect to m_t is

$$b_m(c_t, m_t, h_t)\{-\varphi_t f_{lh}(k_t, l_t h_t) h_t + [\Phi e^{-\rho t} - \Lambda H'(l_{ht})] u_x(c_t, x_t)\} = 0. \tag{63}$$

The first-order conditions with respect to l_t is

$$[\Phi e^{-\rho t} - \Lambda H'(l_{ht})] u_x(c_t, x_t) = 0, \tag{64}$$

which can be substituted into equation (63) to give that

$$b_m(c_t, m_t, h_t)\varphi_t f_{lh}(k_t, l_t h_t) h_t = 0. \tag{65}$$

Case 1. Suppose that $b_m(c_s, m_s, h_s) \neq 0$. Then it would be true that

$$\varphi_t f_{lh}(k_t, l_t h_t) h_t = 0. \tag{66}$$

By equation (19) and the facts that labor and capital are limited, that k_t and h_t are growing at the balanced path growth rate, and that $c_t > 0$ because of Inada conditions on the utility function, so that l_t must be positive, it follows that $f_{lh}(k_t, l_t h_t) > 0$ and $h_t > 0$. And the shadow price of the real resource constraint must be positive since there are positive resources, as in equation (2), and insatiable utility, so that $\varphi_t > 0$. Thus this leads to a contradiction.

Case 2. $b_m(c_s, m_s, h_s) = 0$. Equation (30) implies that $b_m = A_{st} c_t (1/\gamma)[1 - (m_t/c_t)]^{(1/\gamma)-1}/h_t$. With $\gamma \in (0, 1)$, this case is satisfied when $m_t/c_t = a_t = 1$, which is feasible.

Corollary 1 *The Friedman rule of $R_t = 0$ holds at the Ramsey optimum.*

proof. By Lemma 1 $b_m(c_s, m_s, h_s) = 0$. This can be written as $b_m = -b/[(1 - a_t)c_t\gamma] = 0$. And since $(1 - a_t)c_t = d_t$ by equation (11), and $b(c_s, m_s, h_s) = l_{dt}$, then by equation (30), $b_m = -1/(\partial d_t/\partial l_{dt}) = 0$. This is the (negative) inverse of the marginal product of labor in the credit production. The Inada condition on credit production, $\lim_{l_{dt} \to 0} \partial d_t/\partial l_{dt} = \infty$, applies to equation (30) and so implies the satisfaction of the condition $b_m(c_s, m_s, h_s) = 0$ at $l_{dt} = 0$. With no labor in credit production, there is zero credit produced, and this implies by equation (5), (9), (10), and (11) that $a_t = 1$. In turn $a_t = 1$ implies by equations (10) and (50) that $R_t = 0$.

At the Friedman (1969) optimum, the amount of credit services provided (and inflation-tax avoidance) is zero. This in turn implies that Friedman optimum is part of the Ramsey (1927) optimal solution.

6.5 Discussion

In order for the Friedman (1969) rule to be Ramsey (1927) optimal the production of credit must show diminishing returns in terms of the labor input into the credit production function, or, of the banking time. The Inada conditions allow the marginal product to go to infinity as the labor time goes to zero.

The result is not sensitive to non-extreme values of the parameters of the variable cost credit technology. Extreme values of γ and A_d present corner solutions and equilibrium uniqueness problems. If the diminishing returns parameter is given by $\gamma = 1$, then the credit has a constant marginal cost equal to its average cost, and this would be equivalent to a linear production of credit with A_d equal to the constant marginal product of labor. Then there may be no unique equilibrium. If the nominal interest rate coincides with the marginal cost of credit, so that $R_t = w_t/A_d$, then the consumer's equilibrium choice between money and credit is arbitrary. If $R_t < w_t/A_d$, then the consumer uses only money; and if $R_t > w_t/A_d$, the consumer uses only credit and nominal prices are not well-defined. The Friedman (1969) rule would still be first best in these cases, since it would save on resources used in exchange (except when $A_d = \infty$ and credit is free of use as implicitly is the case in Lucas and Stokey (1983)). But if A_d is near zero in the CRS case, it is similar to making credit prohibitively expensive so that the economy is similar to the cash-only Lucas (1980) model. With no viable substitutes to money, the inflation tax then only distorts the consumption margin of goods to leisure and is no worse than a value-added tax on goods purchases.

Money demand is affected by γ and A_d in terms of how interest elastic it is. The effect of γ on the interest elasticity is ambiguous in general, while a higher A_d unambiguously makes the money demand more interest elastic at all inflation rates. Regardless of the particular non-extreme values of γ and A_d, the money demand still exhibits an increasing interest elasticity as the inflation rate rises, as in Cagan (1956), and as is critical in explaining a certain nonlinearity in the inflation-growth effect of the model and as in evidence (Gillman and Kejak 2005b). But the different non-extreme γ and A_d values do not affect the Ramsey (1927) analysis.

6.6 Conclusion

The chapter derives optimal monetary policy with commitment in an endogenous growth economy using an approach based on a price-theoretic description of money and credit. More explicit than the shopping time model in these connections, the consumer uses both credit and money as intermediate goods in producing the household consumption good. The production of credit allows us to relate the conditions for optimality of the Friedman (1969) rule to the underlying credit production technology. The optimality conditions are related directly to conditions for balanced growth (see also Alvarez, Kehoe, and Neumeyer (2004)); the consumption velocity of credit must also be constant on the balanced-growth path. Shifts in the parameters determining the credit velocity, such as in the productivity of credit during financial deregulation, can shift the credit velocity but do not affect the Ramsey analysis.

The chapter gives new intuition to the homogeneity assumption for Ramsey (1927) optimality of the Friedman (1969) rule. Credit use is zero in the optimum and the marginal productivity of credit is infinite at this point, although this is productivity only in avoiding the inflation tax. This bases the proof of the Friedman rule as Ramsey optimal upon the Inada conditions on the production function of credit while giving the intuition that there is no proclivity of the consumer to substitute towards credit at this point, since the interest elasticity of money demand is zero at the optimum.

The money demand implied by the credit technology has been supported with empirical evidence (see Mark and Sul, 2003 and Gillman and Otto, 2002) and is consistent with facets of the inflation experience along the balanced-growth path that also have empirical support (Gillman and Kejak, 2005b, Gillman, Harris, and Matyas, 2004, Gillman and Nakov, 2003).[4] This consistency strengthens the paper's intuition.

Appendix 6.A: derivation of equations

6.A.1 Wealth constraint (28)

From equation (17), add and subtract $r_t\, m_t$ to the RHS and solve for $\dot{k}_t + \dot{m}_t$. With the Fisher equation of interest rates this gives equation (24). Multiply both sides by $e^{-\int_0^t \tilde{r}_s ds}$ and integrate both sides over the infinite horizon:

$$e^{-\int_0^t \tilde{r}_s ds}[\dot{W}_t - \tilde{r}_t W_t]dt = e^{-\int_0^t \tilde{r}_s ds}[c_t + R_t m_t - \tilde{w}_t h_t l_t]dt. \tag{67}$$

The LHS of this equation can be written as

$$\begin{aligned}\int_0^\infty (e^{-\int_0^t \tilde{r}_s ds} W_t)dt &= \lim_{t\to\infty} (e^{-\int_0^t \tilde{r}_s ds} W_t) - e^{-\int_0^0 \tilde{r}_s ds} W_0 \\ &= \lim_{t\to\infty} e^{-\int_0^t \tilde{r}_s ds} m_t + \lim_{t\to\infty} e^{-\int_0^t \tilde{r}_s ds} k_t - (k_0 + m_0 + B_0/P_0). \end{aligned}\tag{68}$$

Imposing the transversality conditions (25) and (26) gives the wealth constraint (28).

6.A.2 Implementability conditions

Equations (33), (34) and (35) imply that

$$w_t h_t = \frac{e^{-\rho t} u_x(t)}{e^{-\rho t}[u_c(t) - u_x(t) b_c(t)]}, \tag{69}$$

which gives the equation (54).

Equation (34), (35), and (37) imply equation (55).

Equation (33), (34) and (36) imply that

$$\lambda = \frac{e^{-\rho t} u_x(t)}{e^{-\int_0^t \tilde{r}_s ds} w_t h_t}. \tag{70}$$

At time 0, the constant λ is given by

$$\lambda = \frac{u_x(0)}{w_0 h_0}. \tag{71}$$

This implies equation (56):

$$e^{-\int_0^t \tilde{r}_s ds} = \frac{w_0 h_0}{u_x(0)} e^{-\rho t}[u_c(t) - u_x(t) b_c(t)]. \tag{72}$$

To get equation (59), take equation (45) and substitute in for prices from the from equations (54), (55), and (57). Also note that $e^{-\int_t^s \tilde{r}_\xi d\xi} = (e^{-\int_s^t \tilde{r}_\xi d\xi})^{-1}$. Then from equation (56),

$$e^{-\int_t^s \tilde{r}_\xi d\xi} = (e^{-\int_s^t \tilde{r}_\xi d\xi})^{-1} = \frac{u_x(s)}{w_s h_s} e^{-(s-t)\rho} \frac{1}{u_c(t) - u_x(t) b_c(t)}; \tag{73}$$

and this gives that

$$\frac{u_x(t)}{u_c(t) - u_x(t) b_c(t)} = H'(l_{ht}) \int_t^\infty \{\frac{u_x(s)}{w_s h_s} e^{-\rho(s-t)} \frac{w_s h_s[l_s + b_s]}{u_c(t) - u_x(t) b_c(t)}\} ds. \tag{74}$$

And that implies equation (59).

Notes

* Gillman, Max, and Oleg Yerokhin (2005). 'Ramsey-Friedman Optimality in a Banking Time Economy', *Berkeley Electronic Journals in Macroeconomics: Topics*, 5(1), article 16.

1 The framework is developed in Gillman and Kejak (2005b).
2 In Aiyagari, Braun, and Eckstein (1998) the ray is assumed to have a slope not necessarily equal to one; this is crucial for their imposition in equilibrium of an exogenous money demand function.
3 It is a Hicks (1935) suggestion to have the agent "act in part as a bank".
4 See also Aiyagari, Braun, and Eckstein (1998) and Eckstein and Leiderman (1992) who use a Cagan money demand to explain banking and seignoirage respectively.

Part II

Money demand and velocity